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# Symmetrized $\beta$-integers

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**Abstract**

This article deals with $\beta$-numeration systems, which are numeration systems with a non-integral base $\beta > 1$. In this framework, there exist elements which naturally play the role of integers, which are called $\beta$-integers. The set of non-negative $\beta$-integers, denoted by $\mathbb{Z}_\beta^+$, has various equivalent definitions which arise from different points of view. Nevertheless, these definitions may be generalized on negative real numbers in a non-unique way, depending on the chosen framework.

We focus in this article on confluent Parry units. They are the positive roots of the polynomials $X^d - \sum_{i=1}^{d-1} kX^i - 1$, where the integers $d$ and $k$ satisfy $d \geq 2$ and $k \geq 1$. For any of these numbers, we prove that there exists a discrete subset of $\mathbb{R}$, that we denote by $\mathbb{Z}_\beta^s$, which is locally isomorphic to $\mathbb{Z}_\beta^+$, and such that $\mathbb{Z}_\beta^s = -\mathbb{Z}_\beta^s$. Moreover, $\mathbb{Z}_\beta^s$ is a model set and satisfies the inflation property: there exists $\lambda > 1$ such that $\lambda\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$ ($\lambda$ is then called the inflation factor for $\mathbb{Z}_\beta^s$). Finally, we compute inflation factors for $\mathbb{Z}_\beta^s$ of the form $\beta^i$, $i$ being a positive integer.


*Keywords:* $\beta$-numeration; Model set; Substitution; Inflation factor

## 1. Introduction

The formalization and the study of aperiodic structures with a strong form of ordering was performed during the twentieth century, especially concerning subsets of $\mathbb{R}^k$, where $k$ is a positive integer. First, Delone (Delauney) introduced the notion of a *Delone set* $E$ as a set which is relatively dense in $\mathbb{R}^k$ and uniformly discrete, that is, for which there are positive constants $R$ and $r$ such that any ball of radius $R$ contains at least one element of $E$, and any ball of radius $r$ contains at most one element of $E$ [15]. Later, Meyer studied *Meyer sets* [29], that is, sets $E$ such that $E$ and $E - E$ are Delone sets, and *model sets*, a stronger version of Meyer sets [31] which are defined by a cut-and-project scheme; roughly speaking, the elements of a model set are the images under a projection of elements in $\mathbb{R}^k$ that both belong to a lattice of $\mathbb{R}^k$ and a cylinder whose base is a subset of a linear subspace of $\mathbb{R}^k$.

In this article, we consider discrete sets arising from certain numeration systems defined by a non-integral base $\beta > 1$, namely $\beta$-numeration systems, introduced by Rényi [37] and Parry [33]. For any $\beta > 1$, one may define the set of non-negative integers in base $\beta$, or set of non-negative $\beta$-*integers*, denoted by $\mathbb{Z}_\beta^+$. This set consists of non-negative real numbers $x = \sum_{i=0}^{n} v_{-i}\beta^i$, where for all indices $i$ we have $v_i \in \mathcal{A}_\beta = \{0, \ldots, \lceil\beta\rceil - 1\}$, and such that the word $v = v_{-n} \ldots v_0$ satisfies an additional hypothesis known as the *Parry condition*: any of the right truncation of $v$ is

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lexicographically less than $d^*_\beta(1)$, defined as the lexicographically greatest expansion of 1 in base $\beta$ among sequences taking values in $\mathcal{A}_\beta$ that do not consist of finitely many non-zero elements. Indeed, $d^*_\beta(1)$ is an improper expansion which plays the same role as $0.(b-1)^\infty$ in standard numeration systems with integral bases $b$. The set of $\beta$-integers, denoted by $\mathbb{Z}_\beta$, is defined as the set of real numbers whose absolute value is a $\beta$-integer, that is, $\mathbb{Z}_\beta = \mathbb{Z}^+_\beta \cup -\mathbb{Z}^+_\beta$. It is clearly a discrete set which is relatively dense in $\mathbb{R}$. A natural question is to determine for which values of $\beta$ is $\mathbb{Z}_\beta$ a Delone set, a Meyer set or a model set.

For any $\beta > 1$, let $T_\beta$ be the map: $[0, 1] \to [0, 1], x \longmapsto \beta x \mod 1$. Set $K_\beta$ as the $T_\beta$-orbit of 1 without 0. Then the set $K_\beta$ is the set of distances between two consecutive $\beta$-integers (see for instance [3]). We deduce that a necessary condition for $\mathbb{Z}_\beta$ being a Meyer set is that $K_\beta$ is a finite set, and that $\mathbb{Z}_\beta$ is a Delone set if and only if the infimum of $K_\beta$ is positive. Actually, a better relation between the algebraic properties of $\beta$ and the structure of $\mathbb{Z}_\beta$ is given in [12] and [30]: for any Pisot number, $\mathbb{Z}_\beta$ is a Meyer set, and $\mathbb{Z}_\beta$ cannot be a Meyer set when $\beta$ is neither Pisot nor Salem. Note that it is not yet known for which algebraic numbers is $\mathbb{Z}_\beta$ a Delone set.

When $K_\beta$ is finite, $\beta$ is said to be a *Parry number*, and a *simple Parry number* if moreover $T^{(m)}_\beta(1) = 0$. Although the set of Parry numbers is not totally classified from an algebraic point of view, it is known that Parry numbers are Perron numbers [16,26], and that Pisot numbers are Parry numbers [11,39]. Obviously, when $\beta$ is a Parry number, the set $\mathbb{Z}_\beta$ is a Delone set; moreover, one may provide combinatorial and a geometrical frameworks naturally associated with the number system, as follows. Let $m$ denote the number of elements in $K_\beta$. For any $i \in \{1, \ldots, m\}$, set $t_i = T^{(i-1)}_\beta(1)$. The *$\beta$-substitution* [42,18] associated with $\beta$, denoted by $\sigma_\beta$, is defined on the $m$-letter alphabet $\{1, \ldots, m\}$ by:

(1) $\sigma_\beta(i) = 1^{\lfloor \beta t_i \rfloor}(i + 1)$ if $i \neq m$,
(2) $\sigma_\beta(m) = 1^{\lfloor \beta t_m \rfloor}$ if $T^{(m)}_\beta(1) = 0$, or $\sigma_\beta(m) = 1^{\lfloor \beta t_m \rfloor}(i + 1)$ if there exists $i < m$ such that $t_i = t_m$.

Note that for any Parry number $\beta$, the substitution $\sigma_\beta$ has a unique right-sided periodic point, which is a fixed point, that we denote by $\omega_r$. Since $\mathbb{Z}^+_\beta$ is a discrete set, we may consider the tiling of $\mathbb{R}^+$ defined by $\mathbb{Z}^+_\beta$, that is, such that the boundaries of the tiles are the elements of $\mathbb{Z}^+_\beta$. If we code each interval of length $t_i$ by the letter $i$ for any $i \in \{1, \ldots, m\}$, we see that $\omega_r$ is a coding of $\mathbb{Z}^+_\beta$. More precisely, thanks to the Dumont–Thomas algorithm [17], a natural relation between $\mathbb{Z}^+_\beta$ and $\omega_r$ ensues from the formula:

$$n_k = \sum_{i=1}^{m} |\mathrm{pref}_k(\omega_r)|_i T^{(i-1)}_\beta(1),$$

where for any positive integer $k$, $n_k$ denotes the $k$th positive $\beta$-integer; the word $\mathrm{pref}_k(\omega_r)$ denotes the prefix of $\omega_r$ of length $k$ and $|u|_i$ denotes the number of occurrences of the letter $i$ in the word $u$. For more details, see [42].

Suppose now that $\sigma_\beta$ is a Pisot substitution, that is, the characteristic polynomial of its incidence matrix $[M_\sigma]_{i,j} = |\sigma_\beta(j)|_i$ is the minimal polynomial of a Pisot number $\beta$. Then $\mathbb{R}^m$ can be expanded as the direct sum of two stable subspaces for $M_\sigma$: an expanding line $\mathcal{D}$ associated with $\beta$, and a contractive hyperplane $\mathcal{H}$ associated with the algebraic conjugates which differ from $\beta$. There exists a geometrical representation for the substitutive dynamical system associated with $\sigma_\beta$ known as the Rauzy fractal of this substitution, that we denote by $\mathcal{T}$, which is a compact subset of $\mathcal{H}$ obtained as the closure of the projection along $\mathcal{D}$ onto $\mathcal{H}$ of the image of the prefixes of $\omega_r$ under the Parikh map $f : \mathcal{A}^* \to \mathbb{Z}^m, u \longmapsto (|u|_1, \ldots, |u|_m)$.

The elements which play the role of decimal numbers in base $\beta$ define the set $\mathrm{Fin}(\beta) = \cup_{k \in \mathbb{N}} \beta^{-k}\mathbb{Z}_\beta$. When $\mathrm{Fin}(\beta)$ has a ring structure, which holds exactly when $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$, it is said that the *finiteness property*, denoted by $(\mathcal{F})$, holds. Introduced by Frougny and Solomyak in [21], the finiteness property may hold only for bases among Pisot numbers and simple Parry numbers. Whereas not yet fully characterized, classes of numbers satisfying the finiteness property have already been extensively studied; see for instance [23,2,4]. It is proven in [1] that, when $\beta$ is a Pisot unit with property $(\mathcal{F})$, 0 is an inner point of the Rauzy fractal $\mathcal{T}$.

Suppose that $\sigma_\beta$ is a Pisot unimodular substitution, that is, such that $|\det M_\sigma| = 1$, and that the finiteness property holds. Let $\overrightarrow{\pi_\mathcal{D}}$ denote the projection along $\mathcal{H}$ onto $\mathcal{D}$, and $\overrightarrow{\pi_\mathcal{H}}$ denote the projection along $\mathcal{D}$ onto $\mathcal{H}$. Set $\overrightarrow{v_\mathcal{D}} = \overrightarrow{\pi_\mathcal{D}}(\overrightarrow{e_1})$, $\overrightarrow{e_1}$ being the first vector of the canonical basis of $\mathbb{R}^m$, and set $\pi_\mathcal{D}$ as the coordinate map on $\mathcal{D}$, that is, such that $\overrightarrow{\pi_\mathcal{D}}(X) = \pi_\mathcal{D}(X)\overrightarrow{v_\mathcal{D}}$ for any $X \in \mathbb{R}^m$. Then one has $\mathbb{Z}^+_\beta = \{\pi_\mathcal{D}(X) | X \in \mathbb{Z}^m, \overrightarrow{\pi_\mathcal{H}}(X) \in \mathcal{T}, \pi_\mathcal{D}(X) \geqslant 0\}$. In other words, $\mathbb{Z}_\beta^+$ may be (improperly) called a "semi-model" set, in the sense that we use a semi-cylinder instead of a cylinder in these scheme. Note that this property has various equivalent formulations; this is for instance Theorem 8 in [10].

In order to characterize the set $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup -\mathbb{Z}_\beta^+$ as a model set, a necessary condition is that the language $\mathcal{L}_\sigma$ associated with the substitution $\sigma_\beta$, whose words code the patterns of $\mathbb{Z}_\beta$, is stable under the mirror image map. Simple Parry numbers for which this property holds are called *confluent Parry numbers*, or confluent Pisot numbers, since such numbers are actually Pisot numbers. Confluent Pisot numbers were introduced in [19] and studied in [20]. For such numbers, the $\beta$-expansion of 1 is of the form $d_\beta(1) = 0.\lfloor\beta\rfloor^{d-1}k$, where $1 \leqslant k \leqslant \lfloor\beta\rfloor$, and the algebraic degree $d$ of $\beta$ is equal to the number of elements in $K_\beta$.

We have proven in [9] that, for any confluent Pisot unit, the associated Rauzy fractal $\mathcal{T}$ is stable under a central symmetry $s_c : \mathcal{H} \to \mathcal{H}, z \longmapsto 2c - z$, with $c \in \mathcal{T}$; however, the center of symmetry $c$ differs from 0. As a consequence, when the set $\mathbb{Z}_\beta^+$ is a semi-model set whose acceptance window is $\mathcal{T}$, the set $-\mathbb{Z}_\beta$ is a semi-model set defined by the same cut-and-project set and whose acceptance window is $-\mathcal{T} \neq \mathcal{T}$, and $\mathbb{Z}_\beta$ is not a model set. In some sense, this is a consequence of the fact that 0 does not play a natural role for being the center of symmetry of $\mathbb{Z}_\beta$. This argument was already noticed by Hof, Knill and Simon in [22]. However, considering the fact that the Rauzy fractal $\mathcal{T}$ may be stable under a central symmetry on $\mathcal{H}$ for confluent Pisot units let us hope that, if we consider the image of $\mathcal{T}$ under a translation vector adequately chosen, we may obtain an acceptance window $\mathcal{T}^s$ which satisfies $\mathcal{T}^s = -\mathcal{T}^s$, and which defines a model set stable under the map $x \longmapsto -x$.

The aim of this article is to construct and study such a set. We see that, under the assumption that $\beta$ is a confluent Pisot unit, we may define a set $\mathbb{Z}_\beta^s$ such that the following properties are satisfied:

(1) $\mathbb{Z}_\beta^s$ is a model set,
(2) $\mathbb{Z}_\beta^s = -\mathbb{Z}_\beta^s$,
(3) $\mathbb{Z}_\beta^s$ and $\mathbb{Z}_\beta^+$ are locally isomorphic (indistinguishable),
(4) the two-sided word which codes $\mathbb{Z}_\beta^s$ is the fixed point of a substitution.

In Section 2, we introduce the basic definitions and notation needed for our study, and the related frameworks. In Section 3, we define the set $\mathbb{Z}_\beta^s$. We prove that $\mathbb{Z}_\beta^s$ is locally isomorphic to $\mathbb{Z}_\beta^+$, and that $\mathbb{Z}_\beta^s$ is included in a model set whose acceptance window is $\mathcal{T}^s$, a compact subset of $\mathcal{H}$ obtained as the image of $\mathcal{T}$ under a translation. We then study in Section 4 several arithmetical and geometrical properties related to $\mathbb{Z}_\beta^s$ in the unit, non-quadratic case. Notably, we prove the following assertions.

**Theorem.** *Let $\beta$ be a confluent Parry unit of degree $d \geqslant 3$, with $\lfloor\beta\rfloor$ even. Then $\mathcal{T}^s$ is an acceptance window for the model set $\mathbb{Z}_\beta^s$ if and only if $\lfloor\beta\rfloor = 2$ and $d \in \{3, 4\}$.*

**Theorem.** *Let $\beta$ be a confluent Parry unit, with $\lfloor\beta\rfloor$ odd. It is decidable whether $\mathcal{T}^s$ is an acceptance window for the model set $\mathbb{Z}_\beta^s$.*

Section 5 is devoted to the search of *inflation factors* for $\mathbb{Z}_\beta^s$, that is, to the set of real numbers $\{\lambda > 1 | \lambda\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s\}$ that are of the form $\beta^i$, $i$ being a positive integer. The following proposition combines Propositions 5.1 and 5.2.

**Proposition.** *Let $\beta$ be a confluent Parry unit of degree $d$, with $\lfloor\beta\rfloor$ odd. Then:*

(1) *$\beta$ is not an inflation factor for $\mathbb{Z}_\beta^s$,*
(2) *$\beta^{d+1}$ is an inflation factor for $\mathbb{Z}_\beta^s$,*
(3) *if 0 is an inner point of $\mathcal{T}^s$, there exists a positive integer $N$ such that for any $n \geqslant N$, $\beta^n$ is an inflation factor for $\mathbb{Z}_\beta^s$.*

## 2. Definition and notation

**Starting from now on, we assume that $\beta$ is a confluent Pisot unit**. This implies that $\beta$ is a simple Parry number; $\sigma_\beta$ is a $d$-letter Pisot unimodular substitution, where $d$ is the algebraic degree of $\beta$, and the finiteness property $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$ holds [21]; 0 is an inner point of the Rauzy fractal $\mathcal{T}$, and, as a consequence, $\mathcal{T}$ generates a periodic tiling of $\mathcal{H}$, with $(\overrightarrow{\pi_\mathcal{H}}(\overrightarrow{e_i} - \overrightarrow{e_1}))_{i\in\{2,\ldots,d\}}$ as the lattice basis. Furthermore, one has by construction $\pi_\mathcal{D}(\overrightarrow{e_i}) = t_i$ for any $i \in \{1, \ldots, d\}$. We see in Section 2.2 that one may define as well a self-affine aperiodic tiling with finitely many images of $\mathcal{T}$ under similarities. Some of these properties allow us to introduce in the following several simplifications with respect to standard definitions, for instance with those introduced in [31,41]. See [10] for a general survey of related topics.

### 2.1. Generalities

In the following, $\{i, \ldots, j\}$ denotes the set $\{k \in \mathbb{Z} | i \leqslant k \leqslant j\}$ and $\mathbb{N}$ denotes the set of non-negative integers.

*Words*

We refer mainly to [27,34] for the following notation.

Let $m$ be a positive integer. The finite set $\mathcal{A} = \{1, \ldots, m\}$ is called *alphabet*; it consists of *letters*. Endowed with the concatenation map, $\mathcal{A}$ generates a free monoid $\mathcal{A}^*$. The empty word is denoted by $\varepsilon$. A *language* $\mathcal{L}$ is a subset of $\mathcal{A}^*$.

Let $u_1, \ldots, u_n$ be letters in $\mathcal{A}$. The *length* of the word $u = u_1 \ldots u_n$ is $|u| = n$. For any $l \in \mathcal{A}$, we denote by $|u|_l$ the number of occurrences of the letter $l$ in $u$. The *mirror image* of $u$ is $\tilde{u} = u_n \ldots u_1$. When $\tilde{u} = u$, $u$ is said to be a *palindrome*; its *center* is $\varepsilon$ if $n$ is even, or the letter $u_{\frac{n+1}{2}}$ if $n$ is odd. For all $l \in \mathcal{A} \cup \{\varepsilon\}$, we denote by $\mathcal{P}_l$ the set of palindromes of center $l$. The *shift map* on $\mathcal{A}^{\mathbb{Z}}$ is the map $S : (u_i)_{i \in \mathbb{Z}} \longmapsto (u_{i+1})_{i \in \mathbb{Z}}$, which may be naturally defined on $\mathcal{A}^{\mathbb{N}}$. The *circular shift map* is the map defined on $\mathcal{A}^*$ by $S_c : u_1 \ldots u_n \longmapsto u_2 \ldots u_n u_1$.

Let $w_r$ be a right-sided sequence. For any $n \in \mathbb{N}$, we denote by $\mathrm{pref}_n(w_r)$ the prefix of $w_r$ of length $n$. When $w = w_l.w_r$ is a two-sided sequence, $\mathrm{pref}_n(w)$ denotes the corresponding prefix of $w_r$ if $n \geqslant 0$; we set $\mathrm{pref}_n(w)$ as the suffix of $w_l$ of length $-n$ when $n < 0$.

*Substitutions*

A *substitution* $\sigma$ is a map: $\mathcal{A} \to \mathcal{A}^*$ extended as a morphism for the concatenation map. In this article, we consider Pisot substitutions; due to [14], they are *primitive*: there exists a positive integer $n$ such that, for all $i, j \in \{1, \ldots, m\}$, the word $\sigma^n(i)$ contains at least one occurrence of the letter $j$. Any $\omega \in \mathcal{A}^{\mathbb{Z}}$ is said to be a *$\sigma$-periodic point* (of order $k$) when there exists a positive integer $k$ such that $\sigma^k(\omega) = \omega$. If $\sigma(\omega) = \omega$, $\omega$ is said to be a *$\sigma$-fixed point*. The set of factors that occur in any periodic point of a primitive substitution $\sigma$ is the *substitutive language*, denoted by $\mathcal{L}_\sigma$. The *substitutive dynamical system* $(\mathcal{X}_\sigma, S)$ consists of $\mathcal{X}_\sigma$, the subset of $\mathcal{A}^{\mathbb{Z}}$ whose elements have $\mathcal{L}_\sigma$ as set of factors, and the natural $S$-action on $\mathcal{A}^{\mathbb{Z}}$. If for any $w \in \mathcal{X}_\sigma$, the $S$-orbit of $w$ is dense in $\mathcal{X}_\sigma$, $(\mathcal{X}_\sigma, S)$ is said to be *minimal*. See [35] for more details.

The set of primitive substitutions which generate a language stable under the mirror image map is introduced in [22] as the *class* $(\mathcal{P})$. See [5] for a general study of palindromic properties.

*Tilings*

Let $d$ be a positive integer. A *tile* of $\mathbb{R}^d$ is a non-empty compact set $T \subset \mathbb{R}^d$ such that $\overline{\overset{o}{T}} = T$. A *tiling* $\Lambda$ of $E \subset \mathbb{R}^d$ is a collection of tiles such that any compact $K \subset E$ can be covered by finitely many tiles of $\Lambda$, and such that any intersection of distinct tiles has a zero-Lebesgue measure. A *pattern* is a finite connected collection of tiles in a tiling. Two tilings are said to be *locally isomorphic* (or *locally indistinguishable*, see [8]) if they have the same set of patterns.

Let $T$ and $T'$ be two tiles. If there exists $t \in \mathbb{R}^d$ such that $T + t = T'$, $T$ and $T'$ are said to be *equivalent*. The set of tiles which are equivalent to a given tile is a class of equivalence, which is called the *type* of the tile. When there are only finitely many different types of tiles in a tiling $\Lambda$, we define a *coding* of the tiling as the map $\Lambda \to \{1, \ldots, m\}$, where each letter of $\{1, \ldots, m\}$ represents a type of tile.

**Remark 2.1.** When the sequence of real numbers $(x_k)_{k \in \mathbb{Z}}$ is increasing and such that $\{x_{k+1} - x_k | k \in \mathbb{Z}\}$ takes $d$ distinct values, the set $E = \{x_k | k \in \mathbb{Z}\}$ defines a tiling of $\mathbb{R}$: the tiles are the intervals $\{[x_k, x_{k+1}] | k \in \mathbb{Z}\}$, which may be coded using a $d$-letter alphabet. Patterns of $E$ are intervals as well, which may be coded by words.

*Geometrical representation*

Let $\{\alpha_j\}_{j\in\{1,\dots,r+s\}}$ be the set of Galois conjugates which differ from $\beta$ and have a non-negative imaginary part, where $r$ denotes the number of real conjugates which differ from $\beta$ and $s$ denotes the number of non-real conjugates of $\beta$. For convenience, let $J$ denote $\{1,\dots,r+s\}$. For any $j\in J$, we denote by $\tau_j$ the field morphism: $\mathbb{Q}(\beta)\to\mathbb{Q}(\alpha_j)$, $\beta\longmapsto\alpha_j$ if $\alpha_j$ is a real conjugate, $\beta\longmapsto(\operatorname{Re}\alpha_j,\operatorname{Im}\alpha_j)$ otherwise.

The set $\mathbb{R}^d$ may be expanded as the sum of the $M_\sigma$-stable subspaces $\mathcal{D}$ and $\mathcal{H}=\oplus_{j\in J}\mathcal{H}_j$, which are respectively associated with $\beta$ and with the (direct) sum of the eigenspaces whose eigenvalues are $\{\alpha_j\}_{j\in J}$. For any $j\in J$, set $\overrightarrow{\pi_{\mathcal{H}_j}}$ as the projection along $\mathcal{D}\oplus_{i\neq j}\mathcal{H}_i$ onto $\mathcal{H}_j$. We set a basis $(\overrightarrow{v_i})_{i\in\{1,\dots,r+2s\}}$ in $\mathcal{H}$ which is defined by the relations:

(1) $\overrightarrow{\pi_{\mathcal{H}_j}}(\overrightarrow{e_1})=\overrightarrow{v_j}$ for any $j\in\{1,\dots,r\}$,
(2) $\overrightarrow{\pi_{\mathcal{H}_{r+j}}}(\overrightarrow{e_1})=\overrightarrow{v_{r+2j-1}}$ and $\overrightarrow{\pi_{\mathcal{H}_{r+j}}}(M_\sigma\overrightarrow{e_1})=\operatorname{Re}(\alpha_j)\overrightarrow{v_{r+2j-1}}+\operatorname{Im}(\alpha_j)\overrightarrow{v_{r+2j}}$ for any $j\in\{1,\dots,s\}$.

We denote by $\tau$ the map:

$$\tau:\mathbb{Q}(\beta)\to\mathcal{H},\, x\longmapsto\sum_{j=1}^{r}\tau_j(x)\overrightarrow{v_j}+\sum_{j=1}^{s}(\operatorname{Re}\tau_j(x)\overrightarrow{v_{r+2j-1}}+\operatorname{Im}\tau_j(x)\overrightarrow{v_{r+2j}}).$$

**Remark 2.2.** By construction, one has $\overrightarrow{\pi_{\mathcal{H}}}(X)=\tau(\pi_{\mathcal{D}}(X))$ for any $X\in\mathbb{Z}^d$.

*Desubstitution and representation*

The *prefix–suffix automaton* of the substitution $\sigma$, inspired by Rauzy [36] and studied by Canterini and Siegel [13], is defined as $(\mathcal{A},E)$, where $E$ consists of labelled edges $(a,b,(p,l,s))\in\mathcal{A}\times\mathcal{A}\times(\mathcal{A}^*,\mathcal{A},\mathcal{A}^*)$ whenever $\sigma(b)=pas$. The *desubstitution map* $\theta$ is defined by:

$$\theta:\mathcal{X}_\sigma\to\mathcal{X}_\sigma,\,\theta(w)=v \text{ if } w=S^k\sigma(v) \text{ and } k\in\{0,\dots,|\sigma(v_0)|-1\}.$$

For any $w\in\mathcal{X}_\sigma$, we set $\gamma(w)=(p,w_0,s)$ if $\sigma((\theta(w))_0)=pw_0s$ and $w=S^{|p|}\sigma(\theta(w))$. The *prefix–suffix expansion map* $\Gamma$ is defined as follows. For any $w\in\mathcal{X}_\sigma$, $\Gamma(w)$ is defined as the sequence $(p_i,l_i,s_i)_{i\in\mathbb{N}}$ such that one has $w=\dots\sigma^n(p_n)\dots\sigma(p_1)p_0.l_0s_0\sigma(s_1)\dots$. The maps $\theta$, $\gamma$ and $\Gamma$ are well defined and continuous due to [32].

Let $\omega\in\mathcal{X}_\sigma$, with $\Gamma(\omega)=(p_i,l_i,s_i)_{i\in\mathbb{N}}$. The *representation map* is defined as the map $\Phi:\mathcal{X}_\sigma\to\mathcal{T}$, such that $\Phi(\omega)=\sum_{j=1}^{r}(\sum_{i\geqslant0}|p_i|\alpha_j^i\overrightarrow{v_j})+\sum_{j=1}^{s}\left(\sum_{i\geqslant0}|p_i|(\operatorname{Re}\alpha_j^i\overrightarrow{v_{r+2j-1}}+\operatorname{Im}\alpha_j^i\overrightarrow{v_{r+2j}})\right)$. Since $\mathcal{X}_\sigma$ is minimal and $\mathcal{T}$ is compact, $\Phi$ is onto.

*Model sets*

The general definition of a *cut-and-project scheme* requires:

(1) a locally compact topological abelian group $G$, the *internal space*,
(2) $\mathbb{R}^k$, the *physical space*,
(3) a lattice $L\subset\mathbb{R}^k\times G$,
(4) $\overrightarrow{\pi_1}:\mathbb{R}^k\times G\to\mathbb{R}^k$, the first canonical projection, such that the restriction $\overrightarrow{\pi_1}_{|L}$ is one-to-one,
(5) $\overrightarrow{\pi_2}:\mathbb{R}^k\times G\to G$, the second canonical projection, such that $\overrightarrow{\pi_2}(L)$ is dense in $G$.

**Definition 2.3.** Let $\mathcal{U}$ be a relatively compact set of $G$ such that $\overline{\overset{o}{\mathcal{U}}}=\overline{\mathcal{U}}$. The *model set* defined by $\mathcal{U}$, the *window of acceptance* of the model set, is $\Delta(\mathcal{U})=\{\overrightarrow{\pi_1}(x)|x\in L,\overrightarrow{\pi_2}(x)\in\mathcal{U}\}$. When $\partial\mathcal{U}\cap\pi_2(L)=\varnothing$, $\Delta(\mathcal{U})$ is said to be a *regular* model set.

**Remark 2.4.** Model sets may have a non-unique acceptance window. However, due to points (4) and (5) in the definition of a cut-and-project scheme, two acceptance windows $\mathcal{W}_1$ and $\mathcal{W}_2$ of a model set satisfy $\overset{o}{\mathcal{W}_1}=\overset{o}{\mathcal{W}_2}$, hence $\overline{\mathcal{W}_1}=\overline{\mathcal{W}_2}$. As a consequence, there exists at most one tile among the acceptance windows of a given model set.

In this article, we consider cut-and-project schemes associated with confluent Pisot units. Therefore, for a given confluent Pisot unit $\beta$ of degree $d\geqslant2$, we set $k=1$, $G=\mathcal{H}$ and $L=\mathbb{Z}^d$, and we identify $\mathbb{R}^k\times G$ and $\mathbb{R}^d$. Using our previous notation, one has $\overrightarrow{\pi_1}=\overrightarrow{\pi_{\mathcal{D}}}$ and $\overrightarrow{\pi_2}=\overrightarrow{\pi_{\mathcal{H}}}$. Akiyama has proven in [1] that the characteristic properties of a cut-and-project scheme are satisfied in our framework, that is, the restriction $\pi_{\mathcal{D}}:\mathbb{Z}^d\to\mathbb{Z}[\beta]$ is one-to-one, and $\overrightarrow{\pi_{\mathcal{H}}}(\mathbb{Z}^d)$ is dense in $\mathcal{H}$. Note that a model set is defined in our study as a subset of $\mathcal{D}$; by abuse of notation, and up

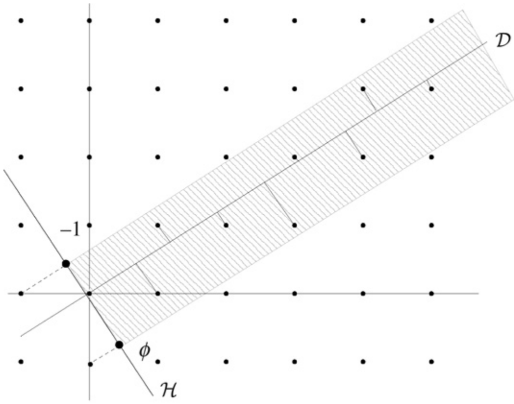


Fig. 1. Cut-and-project scheme for the Fibonacci case.

to the identification $\mathcal{D} \simeq \mathbb{R}$, we may call model set any subset $E$ of $\mathbb{R}$ such that $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in E\}$ is the model set in the associated cut-and-project scheme.

**Example 2.5.** Let $\sigma$ be the *Fibonacci substitution*, defined on the alphabet $\{1, 2\}$ by $\sigma(1) = 12$ and $\sigma(2) = 1$. One has $\pi_{\mathcal{D}}(\overrightarrow{e_1}) = 1$ and $\overrightarrow{\pi_{\mathcal{H}}}(\overrightarrow{e_1}) = \overrightarrow{v_1}$; $\pi_{\mathcal{D}}(\overrightarrow{e_2}) = \phi^{-1}$ and $\overrightarrow{\pi_{\mathcal{H}}}(\overrightarrow{e_2}) = -\phi \overrightarrow{v_1}$. As depicted in Fig. 1, the set $\{X \in \mathbb{Z}^2, \pi_{\mathcal{D}}(X) \geqslant 0, \overrightarrow{\pi_{\mathcal{H}}}(X) \in \mathcal{T}\}$ provides a discrete approximation of the semi-line of equation $y = \phi^{-1}x$. One has $\overrightarrow{\pi_{\mathcal{H}}}(-\overrightarrow{e_1}) = -\overrightarrow{v_1}$ and $\overrightarrow{\pi_{\mathcal{H}}}(-\overrightarrow{e_2}) = \phi \overrightarrow{v_1}$, hence $\overrightarrow{\pi_{\mathcal{H}}}(-\overrightarrow{e_1})$ and $\overrightarrow{\pi_{\mathcal{H}}}(-\overrightarrow{e_2})$ belong to the boundary of $\mathcal{T}$. As a consequence, the model set $\Delta(\mathcal{T}) = \{\overrightarrow{\pi_{\mathcal{D}}}(X) | X \in \mathbb{Z}^2, \overrightarrow{\pi_{\mathcal{H}}}(X) \in \mathcal{T}\}$ is not regular. The sets $\{\lambda \overrightarrow{\pi_{\mathcal{H}}}(\overrightarrow{e_1}) | \lambda \in [-1, \phi[\}$ and $\{\lambda \overrightarrow{\pi_{\mathcal{H}}}(\overrightarrow{e_1}) | \lambda \in ]-1, \phi]\}$ are the respective acceptance windows for the model sets $\{\pi_{\mathcal{D}}(X) | X \in \mathbb{Z}^2, \overrightarrow{\pi_{\mathcal{H}}}(X) \in \mathcal{T}, X \neq -\overrightarrow{e_2}\}$ and $\{\pi_{\mathcal{D}}(X) | X \in \mathbb{Z}^2, \overrightarrow{\pi_{\mathcal{H}}}(X) \in \mathcal{T}, X \neq -\overrightarrow{e_1}\}$. Each of these model sets may be coded by a two-sided periodic point for the Fibonacci substitution, respectively by $(\sigma^2)^\infty(1.1)$ and $(\sigma^2)^\infty(2.1)$.

**Remark 2.6.** In [25], regular model sets are introduced as *generic* model sets, whereas regular model sets are defined as model sets such that $\partial\mathcal{U} \cap \pi_2(L)$ is of (Haar) measure 0.

### 2.2. Parry numeration

Let $x \neq 1$ be a positive real number. The sequence $(v_i)_{i\in\mathbb{Z}}$ taking values in $\mathbb{Z}$ is an *expansion in base $\beta$ of $x$* if $x = \sum_{i\in\mathbb{Z}} v_i \beta^{-i}$. The greatest for the lexicographical order of expansions in base $\beta$ of $x$ taking non-negative values is called *$\beta$-expansion of $x$* and denoted by $d_\beta(x)$; this expansion is computed by the greedy algorithm and satisfies the Parry condition. The set of the factors of the $\beta$-expansions of real numbers is a language denoted by $\mathcal{L}_\beta$. For any $x \in \mathrm{Fin}(\beta)^+$ with $d_\beta(x) = v_{-N} \dots v_0.v_1 \dots v_{N'}$, we define the *$\beta$-integer part of $x$* as $\lfloor x \rfloor_\beta = \sum_{n=0}^{N} v_{-n}\beta^n$, and the *$\beta$-fractional part of $x$* as $\{x\}_\beta = \sum_{n=1}^{N'} v_n \beta^{-n}$.

Let $z \in \mathcal{H}$. The sequence $u = \dots u_0.u_1 \dots u_N 0^\infty$ taking values in $\mathbb{Z}$ with finitely many non-zero values in its right-sided part is an *expansion of $x$ in base $\tau(\beta)$* if $x = \sum_{j=1}^{r} (\sum_{i\in\mathbb{Z}} u_i \alpha_j^i \overrightarrow{v_j}) + \sum_{j=1}^{s} \left( \sum_{i\in\mathbb{Z}} u_i (\mathrm{Re}\, \alpha_j^i \overrightarrow{v_{r+2j-1}} + \mathrm{Im}\, \alpha_j^i \overrightarrow{v_{r+2s}}) \right)$. Note that we may omit the ending 0's in the right-sided part. If moreover $u$ satisfies the Parry condition, $u$ is called a *$\tau(\beta)$-expansion* of $z$. The notion of $\tau(\beta)$-expansion is closely related to the notion of $\alpha$-adic expansions, studied for instance in [6]. Note that, contrary to the uniqueness of the $\beta$-expansion of real numbers, there exist elements in $\mathcal{H}$ which may have distinct $\tau(\beta)$-expansions.

We have proven in [9] that, for any confluent Parry unit $\beta$, there exists $c \in \mathcal{H}$ such that the Rauzy fractal $\mathcal{T}$ is stable under the central symmetry on $\mathcal{H}$ of center $c$. We have computed in [9] the following $\tau(\beta)$-expansion for $c$:

$$c = {}^{\infty}\left(\frac{\lfloor\beta\rfloor}{2}\right). \quad \text{when } \lfloor\beta\rfloor \text{ is even,} \tag{1}$$

$$c = {}^{\infty}\left(\frac{\lfloor\beta\rfloor + 1}{2} 0^{d-1} \frac{\lfloor\beta\rfloor - 1}{2}\right). \quad \text{when } \lfloor\beta\rfloor \text{ is odd.} \tag{2}$$

For convenience, let $\mathcal{L}'_\beta$ denote the set of words in $\mathcal{L}_\beta$ which do not end with 0. Let $w = w_{-N}\dots w_0 w_1 \dots w_{N'} \in \mathcal{L}'_\beta$. The tile $\mathcal{T}_{w_{-N}\dots w_0 . w_1 \dots w_{N'}} \subset \mathcal{H}$ is defined as the closure of $\{\tau(x) | x \in \mathrm{Fin}(\beta)^+, \{\beta^{-N-1}x\}_\beta = \sum_{i=1}^{N+N'+1} w_{i-1-N}\beta^{-i}\}$. Then $\cup_{w \in \mathcal{L}'_\beta} \mathcal{T}_{.w}$ is a tiling of $\mathcal{H}$, and there are $d$ types of tiles $\mathcal{T}_{.w}$ in $\Lambda_\beta$, see [1,42].

**Remark 2.7.** For any $z \in \mathcal{T}$, one has either $z \in \overset{o}{\mathcal{T}}$, or there exists $w \in \mathcal{L}'_\beta$, such that $z \in \partial\mathcal{T} \cap \partial\mathcal{T}_{.w}$.

*Arithmetical automaton*

The notion of *arithmetical automaton* $\mathcal{G} = (V, E)$ is introduced by Rauzy in [36], and can be defined for any $m$-letter unimodular Pisot substitution. Consider the set of states $V \subset \mathbb{Z}^m$, and the set of labelled edges $(X, Y, k) \in V \times V \times \{-\lfloor\beta\rfloor, \dots, \lfloor\beta\rfloor\}$ whenever there exist $X, Y \in V$ such that $Y = M_\sigma X + k\overrightarrow{e_1}$. Then $\mathcal{G}$, the strongly connected component of $(V, E)$ which contains 0, is finite; it is called *arithmetical automaton*. Note that, since the maps $\pi_{\mathcal{D}_{|\mathbb{Z}^d}}$ and $d_\beta$ are one-to-one, any $X \in V$ can be labelled by $d_\beta(\pi_{\mathcal{D}}(X))$.

The arithmetical automaton enables a characterization of elements that belong to $\partial\mathcal{T}$ (see [13]). In particular, thanks to Remark 2.7, the following property holds.

**Proposition 2.8.** *Let $z \in \mathcal{T}$. Then $z \in \partial\mathcal{T}$ if and only if there exists a path in $\mathcal{G}$ labelled by $(u_k - v_k)_{k\leqslant N}$, where $(u_k)_{k\leqslant N}$ and $(v_k)_{k\leqslant N}$ are $\tau(\beta)$-expansions of $z$ such that $u_1 \dots u_N$ are 0's.*

**Proof.** Let $z \in \mathcal{T}$. Due to Remark 2.7, $z \in \partial\mathcal{T}$ if and only if there exists $w \in \mathcal{L}'_\beta$ such that $z$ belongs to $\partial\mathcal{T}$ and to $\partial\mathcal{T}_{.w}$. This is also equivalent to the existence of two $\tau(\beta)$-expansions $\dots u_{-n} \dots u_0$ and $\dots v_{-n} \dots v_0.v_1 \dots v_N$ of $z$, with $w = v_1 \dots v_N$. Since $v_i - u_i \in \{-\lfloor\beta\rfloor, \dots, \lfloor\beta\rfloor\}$ for any $i \in \mathbb{Z}$, $(v_i - u_i)_{i\leqslant N}$ labels a path in $\mathcal{G}$. On the other hand, if there exists a path labelled by $(v_i - u_i)_{i\leqslant N}$ in $\mathcal{G}$, where $(u_i)_{i\leqslant N}$ and $(v_i)_{i\leqslant N}$ are admissible sequences such that $u_1 \dots u_N = 0^N$, then there exists $z \in \mathcal{H}$ such that $\dots u_{-n} \dots u_0$ and $\dots v_{-n} \dots v_0.v_1 \dots v_N$ are $\tau(\beta)$-expansions for $z$, hence $z \in \partial\mathcal{T} \cap \partial\mathcal{T}_{.w}$ with $w = v_1 \dots v_N$. □

There exist modified versions of $\mathcal{G}$ which enable us to determine the neighbour tiles of a given tile, or topological properties like connectedness and simple connectedness for $\mathcal{T}$. See [40,38,24] for more details, and [28] for a detailed study related to the *Tribonacci* numeration system, defined by the positive root of the polynomial $X^3 - X^2 - X - 1$.

## 3. Construction and study of $\mathbb{Z}_\beta^s$

In this section, we are interested in the construction of a discrete subset of $\mathbb{R}$ that may be seen as a tiling (see Remark 2.1), which is stable under the map $x \longmapsto -x$, and whose patterns are coded by a language generated by a $\beta$-substitution. As noticed in the introduction, we need to assume that $\beta$ is a confluent Pisot number to construct such a set. Additionally, we assume that $\beta$ is a unit, since this hypothesis provides additional geometric characterizations.

For any confluent Pisot unit, the following property is satisfied: for any positive integer $n$, there exists a unique palindrome of length $2n$ in $\mathcal{L}_\sigma$ [7]. Hence there exists a unique two-sided sequence $\omega' \in \mathcal{X}_\sigma$ whose left-sided part is the mirror image of its right-sided part, that is, we set $\omega' = \dots u_{-n} \dots u_0.u_1 \dots u_n \dots$ such that, for any positive integer $n$, the word $u_{-n-1} \dots u_0.u_1 \dots u_n$ is a palindrome. We define the set of *symmetrized $\beta$-integers*, that we denote by $\mathbb{Z}_\beta^s$, as the discrete set coded by $\omega'$, where for any $i \in \{1, \dots, d\}$, the letter $i$ codes an interval of length $T_\beta^{(i-1)}(1)$, that is:

$$\mathbb{Z}_\beta^s = \left\{ \sum_{i=1}^{m} |\mathrm{pref}_n(\omega')|_i T_\beta^{(i-1)}(1) | n \in \mathbb{Z} \right\}. \tag{3}$$

### 3.1. Basic properties of $\mathbb{Z}_\beta^s$

Let us recall that $c$ denotes the element in $\mathcal{H}$ such that $\mathcal{T}$ is stable under the symmetry map $s_c : z \longmapsto 2c - z$ defined on $\mathcal{H}$. By definition, one has $d_\beta(T_\beta^{(i-1)}(1)) = 0.\lfloor\beta\rfloor^{d-i}1$ for any $i \in \{1, \dots, d\}$. Hence any $x \in \mathbb{Z}_\beta^s$ is a finite sum of elements of $\mathrm{Fin}(\beta)$ due to (3). As recalled in the beginning of Section 2, the finiteness property $(\mathcal{F})$ holds, hence $\mathbb{Z}_\beta^s \subset \mathrm{Fin}(\beta)$. Moreover, since 0 is an inner point of $\mathcal{T}$, and since the restriction of $M_\sigma$ on $\mathcal{H}$ is contractive, there exists a positive integer $k$ such that $M_\sigma^k(\mathcal{T} - c) \subset \mathcal{T}$. Hence $\mathcal{T} - c$ is covered by finitely many tiles in the self-affine aperiodic tiling generated by $\mathcal{T}$, and there exists a positive integer $k$ such that $\beta^k \mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta$.

**Proposition 3.1.** *For any confluent Parry unit $\beta$, the tilings generated by $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_\beta^s$ are locally isomorphic.*

**Proof.** The tiles coded by a given letter in the tilings associated with $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_\beta^s$ coincide up to translation. Hence an equivalent statement for $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_\beta^s$ being locally isomorphic is that the sets of words which code their patterns are equal.

Since $\omega'$ is defined as the limit of words which belong to $\mathcal{L}_\sigma$, any word which codes a pattern of $\mathbb{Z}_\beta^s$ occurs in $\mathcal{L}_\sigma$. On the other hand, since the language $\mathcal{L}_\sigma$ is generated by a primitive substitution, $\omega_r$ is uniformly recurrent; for any positive integer $k$, there exists a positive integer $n$ such that any word of length $n$ in $\mathcal{L}_\sigma$ contains all the factors in $\mathcal{L}$ whose length is at most $k$. Hence any word in $\mathcal{L}_\sigma$ codes a pattern of $\mathbb{Z}_\beta^s$. $\square$

The local isomorphism implies that the geometrical characterization of $\mathbb{Z}_\beta^s$ and $\mathbb{Z}_\beta^+$ coincide up to translation, that is:

**Corollary 3.2.** *One has $\overline{\tau(\mathbb{Z}_\beta^s)} = \mathcal{T} - c$.*

**Proof.** Due to Proposition 3.1, the $S$-orbit of $\omega'$ is dense in $\mathcal{X}_\sigma$. As a consequence, and since $\mathcal{T}$ is the closure of $\tau(\mathbb{Z}_\beta^+)$, there exists $z \in \mathcal{H}$ such that $\mathcal{T} - z$ is the closure of $\tau(\mathbb{Z}_\beta^s)$.

Let $s_0$ denote the map on $\mathcal{H}$ defined by $s(z) = -z$. Since $\mathcal{T}$ satisfies $\mathcal{T} = s_c(\mathcal{T}) = 2c - \mathcal{T}$, one has $s_0(\mathcal{T} - c) = -\mathcal{T} + c = (2c - \mathcal{T}) - c = s_c(\mathcal{T}) - c = \mathcal{T} - c$. Suppose that $z \neq c$. Since $\overline{\tau(-\mathbb{Z}_\beta^s)} = -\mathcal{T} + z$, $\mathcal{T}$ would be stable under $s_c$ and $s_z$; as a consequence, $\mathcal{T}$ would be stable under the translation map $s_c \circ s_z$, which is absurd since $\mathcal{T}$ is bounded. Hence $z = c$ and the closure of $\tau(\mathbb{Z}_\beta^s)$ is $\mathcal{T} - c$. $\square$

**Notation 3.3.** *We set $\mathcal{T}^s = \overline{\tau(\mathbb{Z}_\beta^s)}$.*

Note that, as a consequence of Corollary 3.2, one has $\Phi(\omega') = c$.

### 3.2. Characterization of regular model sets

In this section, we are interested in determining whether $\mathcal{T}^s$ is an acceptance window for the set $\{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\}$. Corollary 3.2 means that one has $\{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} \subset \Delta(\mathcal{T}^s)$, with $\Delta(\mathcal{T}^s) = \{\overrightarrow{\pi_\mathcal{D}}(X) | X \in \mathbb{Z}^d, \overrightarrow{\pi_\mathcal{H}}(X) \in \mathcal{T}^s\}$. Recall that, as noticed in Remark 2.4, there may exist distinct acceptance windows for $\mathbb{Z}_\beta^s$; however, if $W$ is an acceptance window for $\mathbb{Z}_\beta^s$, one has $\overset{o}{W} = \overset{o}{\mathcal{T}^s}$. As a consequence, any acceptance window $W$ for $\mathbb{Z}_\beta^s$ satisfies $\overset{o}{\mathcal{T}^s} \subset W \subset \mathcal{T}^s$, hence $\Delta(\overset{o}{\mathcal{T}^s}) \subset \{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} \subset \Delta(\mathcal{T}^s)$.

The inequality $\Delta(\overset{o}{\mathcal{T}^s}) \neq \Delta(\mathcal{T}^s)$ corresponds to the case where the boundary of $\mathcal{T}^s$ intersects $\overrightarrow{\pi_\mathcal{H}}(\mathbb{Z}^d)$, which means that $\{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\}$ is not a regular model set. In this case, setting $\mathcal{T}'$ as $\mathcal{T}^s$ minus $\partial \mathcal{T}^s \cap \overrightarrow{\pi_\mathcal{H}}(\mathbb{Z}^d)$, the acceptance window $\mathcal{T}''$ of the model set $\mathbb{Z}_\beta^s$ satisfies $\mathcal{T}' \subset \mathcal{T}'' \subset \mathcal{T}^s$. The following proposition provides a characterization to decide whether the model set $\{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\}$ is regular.

**Proposition 3.4.** *Let $\beta$ be a confluent Parry unit. The following assertions are equivalent:*

(1) $\{\lambda \overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$,
(2) *for any $x \in \mathbb{Z}_\beta^s$, $\tau(x)$ is an inner point of $\mathcal{T}^s$.*

**Proof.** Let us prove that (1) implies (2). Suppose that (2) does not hold, that is, there exists $z \in \mathbb{Z}_\beta^s$ such that $\tau(z) \in \partial \mathcal{T}^s$. Since $\mathcal{T}^s$ has a non-empty interior, there exists $x \in \mathbb{Z}_\beta^s$ such that $\tau(x)$ is an inner point of $\mathcal{T}^s$. Without loss of generality, one may choose $z \geqslant 0$ such that $z' = \max_{x \in \mathbb{Z}_\beta^s}\{x < z\}$ satisfies $\tau(z') \in \overset{o}{\mathcal{T}^s}$. There exists $i \in \{1, \ldots, d\}$ such that $z = z' + \pi_\mathcal{D}(\overrightarrow{e_i})$, that is, $\tau(z) = \tau(z') + \overrightarrow{\pi_\mathcal{H}}(\overrightarrow{e_i})$. However, as seen in the beginning of Section 2, $\mathcal{T}$ generates a periodic tiling on $\mathcal{H}$, with $(\overrightarrow{\pi_\mathcal{H}}(\overrightarrow{e_i} - \overrightarrow{e_1}))_{i \in \{2,\ldots,d\}}$ as a lattice basis. Since $\mathcal{T}^s$ is the image of $\mathcal{T}$ under a translation map, $(\overrightarrow{\pi_\mathcal{H}}(\overrightarrow{e_i} - \overrightarrow{e_1}))_{i \in \{2,\ldots,d\}}$ is a lattice basis for the periodic tiling generated by $\mathcal{T}^s$. We deduce that there exists $j \in \{1, \ldots, d\}$, $j \neq i$, such that $\tau(z') + \overrightarrow{\pi_{\mathcal{H}}}(\overrightarrow{e_j})$ belongs to $\partial \mathcal{T}^s$ as well, that is, the model set whose acceptance window is $\mathcal{T}$ contains $z$ and $z + \pi_{\mathcal{D}}(\overrightarrow{e_j} - \overrightarrow{e_i})$. Finally, recall that for any $i \in \{1, \ldots, d\}$ one has $\pi_{\mathcal{D}}(\overrightarrow{e_i}) = t_i$ as seen in the beginning of Section 2. Since the elements $\{t_i\}_{i \in \{1,\ldots,d\}}$ are $\mathbb{Q}$-independent, $x - t_i + t_j$ does not belong to $\mathbb{Z}_\beta^s$, that is, $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}_\beta^s\} \neq \Delta(\mathcal{T}^s)$.

Now, let us prove that (2) implies (1). Suppose that for any $z \in \mathbb{Z}_\beta^s$, $\tau(z)$ is an inner point of $\mathcal{T}^s$. Then $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\overset{o}{\mathcal{T}^s})$ and $\overrightarrow{\pi_{\mathcal{H}}}(\mathbb{Z}^d) \cap \partial \mathcal{T}^s = \varnothing$. Hence $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$. $\square$

**Remark 3.5.** In Section 4, we state using Conjecture 4.13 that, if $\beta$ is a confluent Parry unit such that $\lfloor \beta \rfloor$ is odd, then, for any $x \in \mathbb{Z}_\beta^s$, $\tau(x)$ is an inner point of $\mathcal{T}^s$. Note that, due to Theorem 2 in [1], if 0 is an inner point of $\mathcal{T}$ and if the finiteness property holds, then for any $x \in \mathbb{Z}_\beta^+$, $\tau(x) \in \overset{o}{\mathcal{T}}$; we do not know whether this result could be used to prove Conjecture 4.13.

**Example 3.6.** Let us consider the case of quadratic Pisot units, studied in [12]. In this case, $\beta$ and $\alpha$, the algebraic conjugates of $\beta$, are roots of $X^2 - \lfloor \beta \rfloor X - 1$. One has $\mathcal{H} \simeq \mathbb{R}$; $\mathcal{T} = [-1, \beta]\overrightarrow{v_1}$ and $\overrightarrow{\pi_{\mathcal{H}}}(c) = \frac{\beta-1}{2}\overrightarrow{v_1}$, hence $\mathcal{T}^s = [-\frac{\beta+1}{2}, \frac{\beta+1}{2}]\overrightarrow{v_1}$. The Lebesgue measure of the tiles coded by 1 and 2 are $\mu(1) = 1$ and $\mu(2) = \beta - \lfloor \beta \rfloor$, which belong to $\mathbb{Z}[\beta^{-1}]$. As a consequence, one has $\tau(x) \in ]-\frac{\beta+1}{2}, \frac{\beta+1}{2}[\overrightarrow{v_1}$ for any $x \in \mathbb{Z}_\beta^s$, hence $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ due to Proposition 3.4. Since $\beta$ is a Pisot number, $\alpha \mathcal{T}^s \subset \mathcal{T}^s$ and $\beta \mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$. See also [7] for more details.

## 4. Non-quadratic case

Starting from now on, we assume that $\beta$ is not a quadratic number. We still suppose that $\beta$ is a confluent Parry unit, that is, $d_\beta(1) = 0.\lfloor \beta \rfloor^{d-1}1$, with $d \geqslant 2$. We prove in this section that the two-sided sequence $\omega'$ introduced in Section 3 is the fixed point of a substitution $\sigma'$, which may be explicitly obtained thanks to the $\beta$-substitution $\sigma_\beta$ and the center of symmetry $c$, depending on the parity of $\lfloor \beta \rfloor$. This result and the following folklore lemma will allow us to study in Section 5 inflation factors for $\mathbb{Z}_\beta^s$ of the form $\beta^i$, $i$ being a positive integer.

**Lemma 4.1.** *Let $\sigma$ be a $m$-letter primitive substitution whose dominant eigenvalue is $\lambda$. Let $v$ be a left $M_\sigma$-eigenvector associated with $\lambda$, with positive coordinates. Let $\omega \in \mathcal{X}_\sigma$ be a $\sigma$-fixed point. Then $\lambda$ is an inflation factor for $E = \{\sum_{i=1}^m |\mathrm{pref}_n(\omega)|_i v_i \mid n \in \mathbb{Z}\}$.*

**Remark 4.2.** Under the hypotheses of Lemma 4.1, $\lambda$ is a Perron number, which implies that $v$ may be chosen with positive coordinates. In the framework of $\beta$-substitutions, and with the notation defined in the introduction, we set $v_i = t_i$ for any $i \in \{1, \ldots, d\}$.

The following notation is useful for the following sections.

**Notation 4.3.** *Let $u$ and $v$ be two-sided sequences taking values in $\mathbb{Z}$. We denote by $u \oplus_d v$ the digit-by-digit addition of $u$ and $v$. We denote by $-u$ the sequence such that $u \oplus_d (-u)$ takes only the value 0. When $u$ has infinitely many consecutive occurrences of zeros, we may omit these consecutive occurrences, and denote it as a one-sided sequence or a word.*

### 4.1. Case $\lfloor \beta \rfloor$ even

Suppose that $\lfloor \beta \rfloor$ is even. Let $\sigma'_\beta = S_c^{\frac{\lfloor \beta \rfloor}{2}} \circ \sigma_\beta$, that is, $\sigma'_\beta(i) = 1^{\frac{\lfloor \beta \rfloor}{2}}(i+1)1^{\frac{\lfloor \beta \rfloor}{2}}$ for all $i \in \{1, \ldots, d-1\}$ and $\sigma_\beta(d) = 1$. Then, for any $i \in \{1, \ldots, d\}$, the word $\sigma'_\beta(i)$ is a palindrome of center $i+1$; since the set of palindromes of even length is stable under $\sigma'_\beta$, we deduce that $\omega' = \sigma'^{\infty}_\beta(1.1)$. As a consequence of Lemma 4.1, $\beta$ is an inflation factor for $\mathbb{Z}_\beta^s$.

**Example 4.4.** Let $\beta$ be the positive root of $X^3 - 2X^2 - 2X - 1$. Then $\mathbb{Z}_\beta^s$ is coded by $\sigma'^{\infty}_\beta(1.1)$, with $\sigma'_\beta(1) = 121$, $\sigma'_\beta(2) = 131$, $\sigma'_\beta(3) = 1$; one has $\beta \mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$.

The following theorem characterizes the sets $\mathbb{Z}_\beta^s$ for which $\{\lambda \overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ holds.

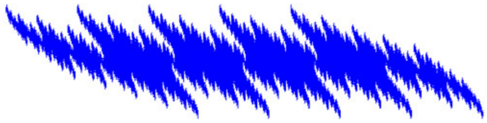

Fig. 2. Rauzy fractal for the numeration system defined by $d_\beta(1) = 0.441$.

**Theorem 4.5.** *Let $\beta$ be a confluent Parry unit of degree $d \geqslant 3$, with $\lfloor\beta\rfloor$ even. Then $\{\lambda\overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}^s_\beta\} = \Delta(\mathcal{T}^s)$ if and only if $\lfloor\beta\rfloor = 2$ and $d \in \{3, 4\}$.*

**Proof.** Let us recall that $(u_i)_{i\in\mathbb{Z}} =^\infty (\frac{\lfloor\beta\rfloor}{2})$ is a $\tau(\beta)$-expansion of $c$, see (1). First, assume $\lfloor\beta\rfloor \geqslant 4$. Then $w = (-1)\lfloor\beta\rfloor^{d-1}1$ is an expansion of 0 in base $\beta$. Let $w' = (-w)0 \oplus_d w = 1(-\lfloor\beta\rfloor - 1)0^{d-2}(\lfloor\beta\rfloor - 1)1$. Set $w'' = \bigoplus_{d, i\in\mathbb{N}} (w')0^{i(d-1)}$. Then $w''$ is an expansion of 0 in base $\tau(\beta)$, and one has $w'' =^\infty (1(-2)10^{d-4})0(\lfloor\beta\rfloor - 1)1$, if $d \geqslant 4$, or $w'' =^\infty (2(-2))1(\lfloor\beta\rfloor - 1)1$, if $d = 3$. In the first case, $(v_i)_{i\in\mathbb{Z}} =^\infty (1(-2)10^{d-4}).0(\lfloor\beta\rfloor - 1)1$ is an expansion of 0 in base $\tau(\beta)$, and $(u_i + v_i)_{i\in\mathbb{Z}}$ is an expansion of $c$ in base $\tau(\beta)$. Moreover, for any non-negative integer $i$, one has $\lfloor\beta\rfloor \geqslant \frac{\lfloor\beta\rfloor}{2} + 1 \geqslant u_i + v_i \geqslant \frac{\lfloor\beta\rfloor}{2} - 2 \geqslant 0$. Hence $u_i + v_i \in \mathcal{A}_\beta$ and $(u_i + v_i)_{i\in\mathbb{Z}}$ satisfies the admissibility condition, that is, $(u_i + v_i)_{i\in\mathbb{Z}}$ is a $\tau(\beta)$-expansion of $c$. This means that $c$ belongs to the tile $\mathcal{T}_{.0(\lfloor\beta\rfloor-1)1}$. According to Remark 2.7, we deduce that $c \notin \overset{o}{\mathcal{T}}$, hence $\{\lambda\overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}^s_\beta\} \neq \Delta(\mathcal{T}^s)$ due to Proposition 3.4. If $d = 3$, then $(v_i)_{i\in\mathbb{Z}} =^\infty (2(-2)).1(\lfloor\beta\rfloor - 1)1$ is an expansion of 0 in base $\tau(\beta)$; we prove similarly that $(u_i + v_i)_{i\in\mathbb{Z}}$ is a $\tau(\beta)$-expansion of $c$, hence $c \in \mathcal{T}_{.1(\lfloor\beta\rfloor-1)1}$ and $c \notin \overset{o}{\mathcal{T}}$.

Now, suppose that $\lfloor\beta\rfloor = 2$ and $d \geqslant 4$. Let $w = (-1)2^{d-1}1$ and $w' = (-w)00 \oplus_d w = 1(-2)(-3)0^{d-3}121$, which is an expansion of 0 in base $\tau(\beta)$. Then $w'' = \bigoplus_{d, i\in\mathbb{N}} w'0^{i(d-1)} =^\infty (1(-1)(-1)10^{d-4})0121$ and $(v_i)_{i\in\mathbb{Z}} =^\infty (1(-1)(-1)10^{d-4}).0121$ are expansions of 0 in base $\tau(\beta)$. Since $(u_i)_{i\in\mathbb{Z}} =^\infty 1$, we deduce that $(u_i + v_i)_{i\in\mathbb{Z}} =^\infty (20021^{d-4}).0121$ is a $\tau(\beta)$-expansion of $c$, that is, $c \in \mathcal{T}_{.0121}$; one has $c \notin \overset{o}{\mathcal{T}}$ and $\{\lambda\overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}^s_\beta\} \neq \Delta(\mathcal{T}^s)$ using the same arguments as in the case $\lfloor\beta\rfloor \geqslant 4$.

Finally, we explicitly compute the arithmetic automaton for the numeration systems defined by $d_\beta(1) = 0.221$ and $d_\beta(1) = 0.2221$. In both cases, we check by pure computation that there does not exist a path in the arithmetic automaton which satisfies the hypothesis of Proposition 2.8. This implies that for any $z \in \mathcal{T}^s$, $\tau(z)$ is an inner point of $\mathcal{T}^s$, hence $\{\lambda\overrightarrow{v_{\mathcal{D}}} | \lambda \in \mathbb{Z}^s_\beta\} = \Delta(\mathcal{T}^s)$. □

**Example 4.6.** Let us consider the numeration system defined by $d_\beta(1) = 0.441$, depicted in Fig. 2. The sequence $^\infty(2(-2)).131$ is an expansion of 0 in base $\tau(\beta)$. There are three distinct $\tau(\beta)$-expansions of $c$, namely $^\infty 2$, $^\infty(40).131$ and $^\infty(40)3.31$. Hence $c$ belongs to the tiles $\mathcal{T}$, $\mathcal{T}_{.131}$ and $\mathcal{T}_{.31}$, which implies $0 \in \partial\mathcal{T}^s$. The smallest positive element $x \in \mathbb{Z}^s_\beta$ such that $\tau(x) \in \overset{o}{\mathcal{T}}$ satisfies $d_\beta(x) = 103.41$; the unique $\tau(\beta)$-expansion of $\tau(x)$ is $^\infty 2331$.

### 4.2. *Case $\lfloor\beta\rfloor$ odd*

Let $\beta$ be a confluent Parry unit of degree $d \geqslant 3$, such that $\lfloor\beta\rfloor$ is odd. We set the integer $N$ and the substitution $\sigma'_\beta$ as:

$$N = \frac{\lfloor\beta\rfloor + 1}{2}|\sigma^d(1)| + \frac{\lfloor\beta\rfloor - 1}{2}, \tag{4}$$

$$\sigma'_\beta = S^N_c \circ \sigma^{d+1}_\beta. \tag{5}$$

Let us recall that, when $l$ is a letter or the empty word, $\mathcal{P}_l$ denotes the set of palindromes whose center is $l$. For convenience, we set $\mathcal{P}_0 = \mathcal{P}_{d+1} = \mathcal{P}_\varepsilon$. Let $h : \mathcal{A}^* \to \mathcal{A}^*$, $v \longmapsto \sigma_\beta(v)a^{\lfloor\beta\rfloor}$. The following lemma is proven in [7,9].

**Lemma 4.7.** *Let $x \in \mathcal{P}_i$, where $i \in \{0, \ldots, d\}$. Then $h(x) \in \mathcal{P}_{i+1}$.*

**Proof.** By definition, the image under $h$ of any letter is a palindrome. Let $x$ be a palindrome of center $i$. There exists $v \in \mathcal{A}^*$ such that $x = \tilde{v}iv$. Since $|h(\tilde{v})| = |h(v)|$, the center of $h(x)$ is the center of $h(i) = 1^{\lfloor\beta\rfloor}(i+1)1^{\lfloor\beta\rfloor}$, which is $i + 1$. □

**Lemma 4.8.** *One has* $\mathcal{L}_{\sigma'} = \mathcal{L}_\sigma$.

**Proof.** Let $l \in \mathcal{L}_\sigma$. If $l \neq d$, then $1^{\lfloor\beta\rfloor}$ is a prefix of $\sigma_\beta(l)$, hence $\sigma_\beta^d(1^{\lfloor\beta\rfloor})$ is a prefix of $\sigma_\beta^{d+1}(l)$. Moreover, since $\sigma_\beta(d) = 1$, then $\sigma_\beta^{d+1}(d) = \sigma_\beta^d(1)$. As a consequence, for any letter $l$, $\sigma_\beta^{d-1}(1)$ is a prefix of $(S_c^{|\sigma_\beta^d(1)|})^{\frac{\lfloor\beta\rfloor+1}{2}} \circ \sigma_\beta^{d+1}(l)$, hence $1^{\frac{\lfloor\beta\rfloor-1}{2}}$ is a prefix of $(S_c^{|\sigma_\beta^d(1)|})^{\frac{\lfloor\beta\rfloor+1}{2}}$ as well. Since the images of the letters of $\mathcal{A}$ under $\sigma_\beta^{d+1}$ have a common prefix of length $N$, the substitutions $\sigma_\beta$ and $S_c^N \circ \sigma_\beta^{d+1}$ generate the same language $\mathcal{L}_\sigma$. $\square$

**Proposition 4.9.** *One has* $\omega' = \sigma'_\beta(1.1)$.

**Proof.** This is exactly Lemma 8.2 in [7]. This can also be seen as a consequence of Proposition 2.3 in [14]. Indeed, the sequence of prefixes in $\Gamma(\omega)$ is $p_i = \varepsilon$ for all non-negative integers $i$. Since a $\tau(\beta)$-expansion of $c$ is ${}^\infty(\frac{\lfloor\beta\rfloor+1}{2}0^{d-1}\frac{\lfloor\beta\rfloor-1}{2})$ as computed in (2), the sequence $(p'_i)_{i\in\mathbb{N}}$ of prefixes in $\Gamma(\omega')$ is periodic of period $d+1$ with $|p'_d|\ldots|p'_0| = \frac{\lfloor\beta\rfloor+1}{2}0^{d-1}\frac{\lfloor\beta\rfloor-1}{2}$. $\square$

**Remark 4.10.** It is stated as a conjecture in [22] that, for any substitutive language $\mathcal{L}_\sigma$ defined on $\{1,\ldots,d\}$ and stable under mirror image, there exist $d+1$ palindromes $\{p_i\}_{i\in\{0,\ldots,d\}}$ and a substitution $\sigma$ defined for any $i \in \{1,\ldots,d\}$ by $\sigma(i) = p_0 p_i$ such that $\mathcal{L}_\sigma = \mathcal{L}$. In the case of $\beta$-substitutions, $\sigma'_\beta$ satisfies these properties, with $p_0 = \varepsilon$.

**Example 4.11.** Let $\beta$ be the Tribonacci number. Remind that this number is the positive root of the polynomial $X^3 - X^2 - X - 1$; $\sigma_\beta$ is then the 3-letter substitution defined by $\sigma_\beta(1) = 12$, $\sigma_\beta(2) = 13$ and $\sigma_\beta(3) = 1$. With our notation, one has $d = 3$ and $n = 7$; $\sigma'_\beta(1) = 1213121213121$, $\sigma'_\beta(2) = 12131213121$ and $\sigma'_\beta(3) = 1213121$.

**Theorem 4.12.** *Let $\beta$ be a non-quadratic confluent Parry unit such that $\lfloor\beta\rfloor$ is odd. It is effectively computable to determine whether* $\{\lambda\overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ *holds.*

**Proof.** Due to Proposition 3.4, the relation $\{\lambda\overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ holds if and only if the image of any element in $\mathbb{Z}_\beta^s$ under $\tau$ is an inner point of $\mathcal{T}^s$. Due to Proposition 2.8, we may check this condition by looking at paths in the associated arithmetical automaton, as follows.

Let us recall that $\omega_r$ and $\omega'_r$ denote the right-sided sequences that are respectively fixed points of $\sigma_\beta$ and $\sigma'_\beta$. The sequence $\omega'_r$ does not belong to the $S$-orbit of $\omega_r$. Hence, due to [13], the prefix–suffix expansions of $\omega'$ and $S^k(\omega')$ differ in only finitely many elements for any $k \in \mathbb{Z}$. Moreover, since $c$ has an ultimately periodic $\tau(\beta)$-expansion, $\omega'$ has an ultimately periodic prefix–suffix expansion; the periodic parts of $\omega'$ and $S^k(\omega')$ coincide. As a consequence, we have to check whether there exists a path of the form $(w_n - v_n)_{n\in\mathbb{N}}$ in the associated automaton, where $(v_n)_{n\in\mathbb{N}}$ is a $\tau(\beta)$-expansion whose periodic part coincides with the periodic part of the $\tau(\beta)$-expansion of $c$, and $(w_n)_{n\in\mathbb{N}}$ is a $\tau(\beta)$-expansion.

Since the arithmetical automaton is finite, $(w_n - v_n)_{n\in\mathbb{N}}$ may be chosen as a loop, that is, $(w_n)_{n\in\mathbb{N}}$ may be chosen as periodic; the length of the periodic part of $(w_n - v_n)_{n\in\mathbb{N}}$ divides the length of the periodic part of $(v_n)_{n\in\mathbb{N}}$, which is $d+1$ due to (2). The arithmetical automaton contains finitely many loops whose lengths are less than or equal to $d+1$, hence we may compute all such loops and determine whether one has $\{\lambda\overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$. $\square$

At the moment, we do not know any example of a confluent Parry unit $\beta$ with $\lfloor\beta\rfloor$ odd and for which $\{\lambda\overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ does not hold. Hence the following conjecture:

**Conjecture 4.13.** *One has* $\{\lambda\overrightarrow{v_\mathcal{D}} | \lambda \in \mathbb{Z}_\beta^s\} = \Delta(\mathcal{T}^s)$ *for any confluent Parry unit $\beta$ such that $\lfloor\beta\rfloor$ is odd.*

## 5. Inflation factors for $\mathbb{Z}_\beta^s$

The inflation property is characteristic of fractal structures, and naturally appears in substitutive dynamical systems. This is why we are interested in this section in the set of inflation factors for $\mathbb{Z}_\beta$. Note that the set of inflation factors for $\mathbb{Z}_\beta^s$ is obviously a monoid for the multiplication. In particular, we are interested in the set of integers $i$ for which $\beta^i\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$ holds. As seen in Section 3.1, there exists a positive integer $k$ such that $\beta^k\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta$; as a consequence, if 0 is an inner point of $\pi_\mathcal{H}(\mathcal{T}^s)$, then there exists a positive integer $n$ such that $\beta^i\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$ for any integer $i \geqslant n$.

When $\lfloor\beta\rfloor$ is even, one has $\sigma'_\beta = S_c^{\frac{\lfloor\beta\rfloor}{2}} \circ \sigma_\beta$ as computed in (1). As a consequence, and due to Lemma 4.1, if $\lfloor\beta\rfloor$ is even, then for any positive integer $i$, $\beta^i$ is an inflation factor for $\mathbb{Z}_\beta$.

### 5.1. Case $\lfloor\beta\rfloor$ odd

In this section, we assume that $\beta$ is a confluent Parry unit of degree $d$, such that $\lfloor\beta\rfloor$ is odd. Then, one has the following results.

**Proposition 5.1.** *The real number $\beta^{d+1}$ is an inflation factor for $\mathbb{Z}_\beta^s$.*

**Proof.** When $\lfloor\beta\rfloor$ is odd, $\sigma'_\beta$ is defined as $S_c^N \circ \sigma_\beta^{d+1}$ (5). Hence the dominant eigenvalue of $\sigma'_\beta$ is $\beta^{d+1}$. Due to Lemma 4.1, this implies $\beta^{d+1}\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$. $\square$

**Proposition 5.2.** *Let $\lfloor\beta\rfloor$ be odd and $d \geqslant 3$. Then $\beta$ is not an inflation factor for $\mathbb{Z}_\beta^s$.*

**Proof.** Let $\lfloor\beta\rfloor$ be odd and $d \geqslant 3$. First, suppose that $\lfloor\beta\rfloor \geqslant 3$. Let $l = \frac{\lfloor\beta\rfloor+1}{2}$. One has $\sigma(d1) = 11^{\lfloor\beta\rfloor}2$. Hence the palindrome of length $\lfloor\beta\rfloor+1$ in $\mathcal{L}_\sigma$ is $1^{\lfloor\beta\rfloor+1}$. Note also that $\sigma_\beta(d1)$ occurs as a factor of $\sigma_\beta^2((d-1)1)$. As a consequence, the word $p = 1^l2(1^{\lfloor\beta\rfloor}2)^{\lfloor\beta\rfloor-1}1^{\lfloor\beta\rfloor}3 = (1^l21^{l-1})^{\lfloor\beta\rfloor}1^l3$ is a prefix of $\omega'_r$.

Let us recall that the Lebesgue measures of the tiles in the tiling associated with $\mathbb{Z}_\beta^+$, or $\mathbb{Z}_\beta^s$ satisfy the following relations: $\mu(1) = 1$, $\mu(2) = T_\beta(1) = \beta - \lfloor\beta\rfloor$ and $\mu(3) = T_\beta^{(2)}(1) = \beta^2 - \lfloor\beta\rfloor\beta - \lfloor\beta\rfloor$. Let $p' = 1^l2$. Since the word $p'$ is a prefix of $\omega'_r$, the real number $x = l\mu(1)+\mu(2) = \beta - \frac{\lfloor\beta\rfloor-1}{2}$ belongs to $\mathbb{Z}_\beta^s$. Moreover, since $p$ is a prefix of $\omega'_r$ as well, $y_1 = \mu((1^l21^{l-1})^{\lfloor\beta\rfloor}1^l) = \lfloor\beta\rfloor\beta + \frac{\lfloor\beta\rfloor+1}{2}$ and $y_2 = \mu((1^l21^{l-1})^{\lfloor\beta\rfloor}1^l3) = \lfloor\beta\rfloor\beta + \frac{\lfloor\beta\rfloor+1}{2} + \beta^2 - \lfloor\beta\rfloor\beta - \lfloor\beta\rfloor = \beta^2 - \frac{\lfloor\beta\rfloor-1}{2}$ belong to $\mathbb{Z}_\beta^s$. Hence $y_1$ and $y_2$ belong to $\mathbb{Z}_\beta^s$, with $|y_2 - y_1| < 1$ and $y_1 < \beta x < y_2$. Finally, one checks that $\mathcal{L}_\sigma$ does not contain any word of the form $\{ij | i \neq 1, j \neq 1\}$. Since intervals coded by the letter $k$ are of length $T_\beta^{(k-1)}(1)$, we obtain that the distance between two $\beta$-integers that are not consecutive is strictly greater than 1. Since $\mathbb{Z}_\beta^s$ is locally isomorphic to $\mathbb{Z}_\beta^+$ (Proposition 3.1), the distance between two elements in $\mathbb{Z}_\beta^s$ that are not consecutive is also greater than 1. This implies that $\beta x \notin \mathbb{Z}_\beta^s$.

Now, suppose that $\lfloor\beta\rfloor = 1$. Let $p = \sigma_\beta^d(1)2$. For any $i \in \{1, \dots, d-1\}$, one has $\sigma_\beta(i) = 1(i+1)$ and $\sigma'_\beta(i) = S_c^{|\sigma^d(1)|} \circ \sigma_\beta^{d+1}(i) = S_c^{|\sigma^d(1)|} \circ \sigma_\beta^d(1(i+1)) = \sigma_\beta^d((i+1)1)$, due to (5).

One checks whether $\sigma^d(2)12$ is a prefix of $\omega'_r$, hence $x = \beta^{d+1} - \beta^d + \beta = \beta^d + \beta - 1 \in \mathbb{Z}_\beta^s$. On the other hand, $\sigma^d(213)$ is a prefix of $\omega'_r$, hence $y_1 = \beta^{d+1} + 1$ and $y_2 = \beta^{d+1} + \beta$ belong to $\mathbb{Z}_\beta^s$, with $y_2 - y_1 = \beta - 1 < 1$. Since $\beta x = \beta(\beta^d + \beta - 1)$, one has $y_1 < \beta x < y_2$, and the same argument as in the case $\lfloor\beta\rfloor \geqslant 3$ proves that $\beta x \notin \mathbb{Z}_\beta^s$. $\square$

### 5.2. The particular case of Tribonacci

We obtain the following result for the particular case of the Tribonacci numeration system, introduced in Example 4.11.

**Proposition 5.3.** *For the Tribonacci numeration system, one has $\beta^k\mathbb{Z}_\beta^s \subset \mathbb{Z}_\beta^s$ if and only if $k \neq 1$.*

**Proof.** As a consequence of Proposition 5.2, $\beta\mathbb{Z}_\beta^s \not\subset \mathbb{Z}_\beta^s$. Let us prove that $\beta^2$ and $\beta^3$ are inflation factors for $\mathbb{Z}_\beta^s$. Since the set of inflation factors for $\mathbb{Z}_\beta^s$ is a monoid for the multiplication, this will imply that for any integer $k \geqslant 2$, $\beta^k$ is an inflation factor for $\mathbb{Z}_\beta^s$.

The Tribonacci case is introduced in Example 4.11. The associated substitution $\sigma'$ is defined by $\sigma'(1) = 1213121213121$, $\sigma'(2) = 12131213121$ and $\sigma'(3) = 1213121$. Let us recall that $f$ denotes the Parikh map, that is, for any word $u$, $f(u) = (|u|_1, |u|_2, |u|_3)$. The images of the letters 1, 2 and 3 under $\sigma'$ are of the form $ABA'C$, $ABA''$ and $AB$ respectively, where $A$, $A'$, $A''$, $B$ and $C$ are words such that $f(A) = f(A') = f(A'') = f(\sigma^2(1))$, $f(B) = f(\sigma^2(2))$ and $f(C) = f(\sigma^2(3))$. As a consequence, $\beta^2$ is an inflation factor for $\mathbb{Z}_\beta^s$. Similarly, the images of 1, 2 and 3 under $\sigma'$ are of the form $AB$, $AC$ and $A$, where $A$, $B$ and $C$ are words such that $f(A) = f(\sigma^3(1))$, $f(B) = f(\sigma^3(2))$ and $f(C) = f(\sigma^3(3))$, hence $\beta^3$ is an inflation factor for $\mathbb{Z}_\beta^s$. $\square$

## Open questions

We believe that a closer study of the arithmetical automaton generated by a confluent Parry unit $\beta$ such that $\lfloor\beta\rfloor$ is odd may provide a proof or a counter-example to Conjecture 4.13.

We do not know whether, for a given confluent Parry unit, there exists $N \in \mathbb{N}$ such that the set of powers of $\beta$ which are inflation factors for $\mathbb{Z}_\beta^s$ is $\{\beta^i, i \geqslant N\}$. If not the case, it is possible to compute the set of inflation factors that are of the form $\beta^i$, $i$ being a positive integer ?

As a consequence of a study performed by Thuswaldner in [43], $\mathcal{T}$ is disk-like for cubic confluent unit Parry numbers. However, we do not know for which confluent unit Parry numbers of higher degree, $\overset{o}{\mathcal{T}}$ is ball-like, and if this property is equivalent to $\partial\mathcal{T}$ being homeomorphic to $\mathbb{S}^{d-1}$.

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